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Solutions for multi-dimensional transient heat conduction with solid body motion

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Abstract

The analytical solution for the problem of transient thermal conduction with solid body movement is developed for an orthotropic parallelepiped. Transformations are used to eliminate the flow terms and the orthotropic dependence. The solution uses two types of Green's functions: one coming from the Laplace transform method and the other from the method of separation of variables. The solution method is powerful because it incorporates internal verification of the numerical results by varying the partition time between the short and long components. An example is given for a multi-dimensional case involving both prescribed heat flux and temperature boundary conditions.

Keywords: Transient; Analysis; Heat conduction; Multi-dimensional; Solid body flow

1. Introduction

The primary motivation of this paper is for the verification of numerical heat transfer codes. This is accomplished by development of a framework for exact solutions for multi-dimensional heat conduction in moving solids; this approach is in contrast to that of generating manufactured solutions [1–3]. One physical application for the transient heat conduction problem involving slug flow in a parallelepiped is in the steel industry where moving solids have transient temperature distributions. Another field for application of this problem is in the study of flow and conduction in porous media.

Methods for solving the heat conduction equation for solid body flow, using a transformation, are given in both [4] and [5]. Ref. [5] also provides a general method for solving transient heat conduction problems using Green's functions, including those addressing solid body motion. The use of these methods for application in verification work is discussed in [6,7]. Solutions for problems involving solid body motion are given in [8], with the emphasis on transient boundary conditions of the first kind (that is, prescribed temperatures). The present paper gives a much more complete derivation of the transformed heat conduction equation, even including orthotropic materials. The present paper also extends this analysis to cases for which there is a boundary condition of the second (prescribed heat flux or Neumann) kind or third (convective condition or Robin) kind on a parallelepiped surface with a boundary condition of the first kind on the opposite face. References for exact solutions for layered slabs are [9–12].

For boundary conditions of second and third kinds, movement of the solid results in changes in the boundary conditions which, in turn, may introduce some unexpected difficulties. One of the difficulties is the transformation of a boundary condition of the second kind to that of the third kind. In our verification studies, two different types of Green's functions are used, one coming from the Laplace transform method and the other from the separation of variables method. Both are used in the solution of a given multi-dimensional heat conduction problem. In general the transformation of

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Nomenclature

B	Biot-like dimensionless number
E	transforming function, Eq. (3a,b)
Fo	Fourier number ($\alpha t/L^2$)
Fo_p	Fourier number corresponding to the partition cotime ($\alpha t_p/L^2$)
$G^S(\ )$	short cotime Green's function
$G^L(\ )$	long cotime Green's function
h_{eff}	effective heat transfer coefficient, W/m^2 °C
IG( )	Green's function integrated over x_i', $i = 1, 2, 3$
k	transformed thermal conductivity, W/m °C
k_i	thermal conductivity in the ith direction, $i = 1, 2, 3$, W/m °C
$K(\ )$	component of short cotime Green's function, m^{-1}
L_1, L_2, L_3	overall length of the body in the x, y, and z directions, respectively, m
M	constant in the heat conduction equation, Eq. (1), s^{-1}
P_i	Peclet-like dimensionless number (U_iL_i/α), $i = 1, 2, 3$
q	heat flux, W/m^2
q_s	prescribed heat flux at a boundary, W/m^2
R_0	dimensionless number, $(Pe/2)^2 - \beta_0^2$
T	temperature, °C
T_s	prescribed surface temperature, °C
U_i	uniform velocity in the x_i direction, $i = 1, 2, 3$, m/s
u	cotime ($t - \tau$), s
x_i	spatial variable in the ith direction, $i = 1, 2, 3$, m

Greek symbols

α	transformed thermal diffusivity, m^2/s
α_i	thermal diffusivity in the ith direction, $i = 1, 2, 3$, m^2/s
β	eigenvalue
κ_j	thermal conductivity defined by Eq. (18), W/m °C
ψ	Transformed temperature variable, Eq. (2)

boundary conditions does not introduce fundamental problems in the solution using Green's functions (GFs) based on the Laplace transform. However, for the separation-of-variables based GFs, all the eigenvalues (for a given direction and boundary conditions of the second and third kinds) are changed for solid body flow in that direction. An anomalous aspect is the introduction of a special eigenvalue which depends upon the value of the Peclet number.

A word of clarification and caution needs to be introduced for this "solid body flow". For flow only in the direction of the boundary conditions of the first kind, no difficulty exists in the physical understanding of the problem; the problem is the one associated with transient slug flow with conduction in the flow direction. For the standard boundary conditions of the second and third kinds in the direction of the flow, the physical picture may not be obvious. In that respect, the solutions given herein come closer to a manufactured solution and are used for verification purposes. Furthermore these solutions have potential for heat transfer with flow in a porous solid but the applications for this are left to others [13, see Example 2].

In the solution of multi-dimensional transient (and actually steady state, as well) heat conduction problems, it is shown in [5–7] that the Green's function time partitioning method is very accurate and efficient; this method uses the two complementary GF forms mentioned above. Either form can be used but using just one can be computationally very inefficient. Furthermore, the accuracy may be adversely affected if just one form is used. Hence both are used and joined at a partition time.

Another extremely important point regarding the use of the two forms of the Green's function is that verification is embedded in the solution method itself. The time at which the short time solution is terminated and the long time solution starts is called the partition time. The partition time can be varied over a range of dimensionless times and still give excellent agreement (to 10 significant figures, for example) in the numerical values for the temperature and heat fluxes. This is an extremely useful and powerful aspect of these solutions. Numerical methods, such as finite differences or control volumes, do not have an analogous feature. Selecting the partition time is discussed below and in [5–7].

An outline of the paper is now given. First, the general describing equation is given for 3D Cartesian coordinates in an orthotropic body. A transformation removes the solid body flow terms and a term proportional to the temperature. The boundary conditions are also transformed. This is followed by the solution of the equation for the transformed variable and implications of the product of the 1D Green's functions for a parallelepiped. Next the 1D GFs are given for boundary conditions of second or third kinds at the heated surface with an isothermal condition (first kind) at the other

surface. Another transformation eliminates the orthotropic terms. Finally results, discussion and examples are given.

2. Problem and transformation equations

The transient heat conduction equation for an orthotropic body with uniform and temperature-independent thermal properties, volume energy generation, uniform velocities in the three directions and a term proportional to the temperature is

$$\sum_{i=1}^{3}\alpha_i\frac{\partial^2 T}{\partial x_i^2}+\frac{g}{C}+MT=\frac{\partial T}{\partial t}+\sum_{i=1}^{3}U_i\frac{\partial T}{\partial x_i},\quad 0<x_i<L_i \tag{1}$$

The coordinates are $x_1=x$, $x_2=y$, $x_3=z$ and the corresponding widths $L_1=L$, $L_2=W$, and $L_3=H$. The MT term can represent a "fin" effect for 1D and 2D analyses, a weak chemical reaction or blood flow in some biological problems. The $\alpha_i=k_i/C$ values are the thermal diffusivities in the x, y and z directions, respectively. The U_i are the velocities in the three different directions, $U_1=U$, $U_2=V$, $U_3=W$; they describe solid body motion and do not change with time. The g is the volumetric energy generation term and C is the uniform volumetric heat capacity.

We now wish to express Eq. (1) in a form which eliminates the $U_i\partial T/\partial x_i$ and MT terms. A derivation is given [5] but it does not treat the orthotropic case; furthermore the below derivative is more direct. Let the temperature be related through $E(x_1,x_2,x_3,t)$ to the new variable ψ by

$$T(x_1,x_2,x_3,t)=E(x_1,x_2,x_3,t)\psi(x_1,x_2,x_3,t) \tag{2}$$

$$E(x_1,x_2,x_3,t)=\mathrm{e}^{Mt}\prod_{i=1}^{3}E_{x_i}(x_i,t),$$

$$E_{x_i}(x_i,t)=\mathrm{e}^{\frac{U_i x_i}{2\alpha_i}-\frac{U_i^2 t}{4\alpha_i}} \tag{3a, b}$$

The following derivatives are needed:

$$\frac{\partial T}{\partial x_i}=E\frac{\partial\psi}{\partial x_i}+\frac{\partial E}{\partial x_i}\psi=E\left[\frac{\partial\psi}{\partial x_i}+\frac{U_i}{2\alpha_i}\psi\right] \tag{4a}$$

$$\frac{\partial^2 T}{\partial x_i^2}=E\frac{\partial^2\psi}{\partial x_i^2}+2\frac{\partial E}{\partial x_i}\frac{\partial\psi}{\partial x_i}+\frac{\partial^2 E}{\partial x_i^2}\psi$$

$$=E\left[\frac{\partial^2\psi}{\partial x_i^2}+\frac{U_i}{\alpha_i}\frac{\partial\psi}{\partial x_i}+\left(\frac{U_i}{2\alpha_i}\right)^2\psi\right] \tag{4b}$$

$$\frac{\partial T}{\partial t}=E\frac{\partial\psi}{\partial t}+\frac{\partial E}{\partial t}\psi=E\left[\frac{\partial\psi}{\partial t}+M\psi-\psi\sum_{i=1}^{3}\frac{U_i^2}{4\alpha_i}\right] \tag{4c}$$

Introducing Eqs. (4) into Eq. (1) gives

$$E\sum_{i=1}^{3}\left[\alpha_i\frac{\partial^2\psi}{\partial x_i^2}+U_i\frac{\partial\psi}{\partial x_i}+\frac{U_i^2}{4\alpha_i}\psi\right]+\frac{g}{C}+EM\psi$$
$$=E\left[\frac{\partial\psi}{\partial t}+M\psi-\sum_{i=1}^{3}\frac{U_i^2}{4\alpha_i}\psi+\sum_{i=1}^{3}\left(U_i\frac{\partial\psi}{\partial x_i}+\frac{U_i^2}{2\alpha_i}\psi\right)\right]$$
$$=E\left[\frac{\partial\psi}{\partial t}+M\psi+\sum_{i=1}^{3}\left(U_i\frac{\partial\psi}{\partial x_i}+\frac{U_i^2}{4\alpha_i}\psi\right)\right] \tag{5}$$

Further simplifying this equation gives the desired result of

$$\sum_{i=1}^{3}\alpha_i\frac{\partial^2\psi}{\partial x_i^2}+\frac{g}{EC}=\frac{\partial\psi}{\partial t},$$
$$\frac{1}{E}=E^{-1}=\mathrm{e}^{-Mt}\prod_{i=1}^{3}\mathrm{e}^{-\frac{U_i x_i}{2\alpha_i}+\frac{U_i^2 t}{4\alpha_i}} \tag{6a, b}$$

This equation is further simplified to remove the orthotropic terms starting at Eq. (20a).

The initial and boundary conditions in terms of temperature are now considered. The initial condition is

$$T(x_1,x_2,x_3,0)=F(x_1,x_2,x_3) \tag{7}$$

and the standard boundary conditions of the first, second and third kinds at $x_1=0$ and L_1 are

$$T(0,x_2,x_3,t)=f_1(x_2,x_3,t),$$
$$T(L_1,x_2,x_3,t)=f_2(x_2,x_3,t) \tag{8a, b}$$

$$-k_1\frac{\partial T}{\partial x_1}(0,x_2,x_3,t)=f_1(x_2,x_3,t),$$
$$k_1\frac{\partial T}{\partial x_1}(L_1,x_2,x_3,t)=f_2(x_2,x_3,t) \tag{9a, b}$$

$$-k_1\frac{\partial T}{\partial x_1}(0,x_2,x_3,t)=f_1(x_2,x_3,t)-h_1T(0,x_2,x_3,t)$$
$$k_1\frac{\partial T}{\partial x_1}(L_1,x_2,x_3,t)=f_2(x_2,x_3,t)-h_2T(L_1,x_2,x_3,t) \tag{10a, b}$$

The source terms in Eqs. (8a,b) are temperatures; in Eqs. (9a,b) are heat fluxes; and in Eqs. (10a,b) are usually the heat transfer coefficient times the ambient temperature. (Notice that the flow is not explicitly included in the boundary conditions of the second and third kinds. Without this explicit inclusion, the problem formulation moves closer to that of manufactured solutions [1].) To make the notation more compact, let the source term be denoted $f_i(r_i,t)$ where i goes from 1 to 6 for a parallelepiped and the r_i values are

$$r=x_1,x_2,x_3\quad r_1=0,x_2,x_3$$
$$r_2=L_1,x_2,x_3\quad r_3=x_1,0,x_3$$
$$r_4=x_1,L_2,x_3\quad r_5=x_1,x_2,0\quad r_6=x_1,x_2,L_3 \tag{11}$$

Now the initial and boundary conditions for ψ can be written using Eq. (2) as

$$\psi(r,0) = E^{-1}(r,0)F(r) \quad \text{initial condition} \tag{12}$$

$$\psi(r_j,t) = E^{-1}(r_j,t)f_j(r_j,t) \quad \text{first kind b.c.} \tag{13}$$

$$\kappa_j \frac{\partial \psi}{\partial n_j}(r_j,t) = E^{-1}(r_j,t)f_j(r_j,t) + \frac{CU_j}{2}\psi(r_j,t) \quad \text{second kind} \tag{14}$$

$$\kappa_j \frac{\partial \psi}{\partial n_j}(r_j,t) = E^{-1}(r_j,t)f_j(r_j,t) - h_j\psi(r_j,t) + \frac{CU_j}{2}\psi(r_j,t) = E^{-1}(r_j,t)f_j(r_j,t) - \left[h_j - \frac{CU_j}{2}\right]\psi(r_j,t), \quad \text{third kind} \tag{15}$$

The r_j values are those given in Eq. (11) and the f_j and h_j correspond to the same surfaces. However, the n_j, U_j and κ_j variables are

$$n_1 = -x, \quad n_2 = x, \quad n_3 = -y, \quad n_4 = y, \quad n_5 = -z, \quad n_6 = z \tag{16}$$

$$U_{j=1} = U_{j=2} = U, \quad U_{j=3} = U_{j=4} = V, \quad U_{j=5} = U_{j=6} = W \tag{17}$$

$$\kappa_1 = \kappa_2 = k_1, \quad \kappa_3 = \kappa_4 = k_2, \quad \kappa_5 = \kappa_6 = k_3 \tag{18}$$

Notice that the boundary condition of the third kind, given by Eq. (15), suggests that we can write

$$h_{j,\text{eff}} = h_j - \frac{CU_j}{2}, \quad j = 1,2,3 \tag{19a}$$

It is important to note that the boundary condition of the second kind on T, Eq. (9), goes to the third kind in terms of ψ. This is shown by Eq. (14) which might be considered to have an "effective heat transfer coefficient" of $-CU_j/2$; note that this coefficient can be either negative or positive because the flow velocity can be positive (positive x_i direction) or negative (negative x_i direction). Because Eq. (19a) can produce a negative value, it may not be the best interpretation to state that Eq. (19a) yields an effective heat transfer coefficient. It is convenient to write Eq. (19a) in the dimensionless form

$$B_{j,\text{eff}} \equiv \frac{h_{j,\text{eff}}L_j}{\kappa_j} = B_j - \frac{1}{2}P_j, \quad B_j \equiv \frac{h_jL_j}{\kappa_j}, \quad P_j \equiv \frac{U_jL_j}{\kappa_j/C} \tag{19b}$$

We have purposefully not labeled these dimensionless quantities *Bi* and *Pe* because they are not the usual Biot and Peclet numbers for several reasons including the possible negative values allowed in Eq. (19b).

At this point another transformation is introduced to remove the orthotropic nature of Eq. (1); for three-dimensional cases the transformed thermal conductivity and diffusivity are defined by

$$k \equiv (k_1k_2k_3)^{1/3}, \quad \alpha \equiv (\alpha_1\alpha_2\alpha_3)^{1/3} \tag{20a}$$

Let the transformed coordinates be defined by

$$x_i^* \equiv x_i\left(\frac{k}{k_i}\right)^{1/2}, \quad i = 1,2,3 \tag{20b}$$

Then the heat conduction equation for ψ given by Eq. (6) can be re-written as

$$\alpha\sum_{i=1}^{3}\frac{\partial^2\psi}{\partial(x_i^*)^2} + \frac{g}{EC} = \frac{\partial\psi}{\partial t} \tag{21a}$$

This is the final transformed heat conduction equation. The initial condition and boundary condition of the first kind, given respectively by Eqs. (12) and (13), are unchanged by these transformations. Boundary condition of the second kind, Eq. (14), can now be written as

$$k\frac{\partial\psi}{\partial n_j^*}(r_j^*,t) = E^{-1}(r_j^*,t)f_j^*(r_j^*,t) + \frac{CU_j^*}{2}\psi(r_j^*,t) \tag{21b}$$

where for $j = 1,2,\ldots,6$

$$n_j^* = n_j\left(\frac{k}{\kappa_j}\right)^{1/2}, \quad U_j^* = U_j\left(\frac{k}{\kappa_j}\right)^{1/2}, \quad f_j^* = f_j\left(\frac{k}{\kappa_j}\right)^{1/2}, \quad h_j^* = h_j\left(\frac{k}{\kappa_j}\right)^{1/2} \tag{21c}$$

The boundary condition of the third kind, Eq. (15), is similarly changed. The "Biot" and "Peclet" numbers given in Eq. (19b) can also be written as

$$B_j^* = \frac{h_j^*L_j^*}{k}, \quad P_j^* = \frac{U_j^*L_j^*}{k/C} = \frac{U_j^*L_j^*}{\alpha} \tag{21d}$$

For convenience, the asterisk superscript is omitted below.

3. Green's function solution equation for the transformed variable, ψ

The solution for ψ in terms of Green's functions [5] is given by

$$\psi(r,t) = \psi_{\text{i.c.}}(r,t) + \psi_{\text{g}}(r,t) + \psi_{\text{b.c.}}(r,t) \tag{22}$$

which has terms for the initial condition, volume energy generation and boundary conditions, respectively. The initial condition term [5] becomes

$$\psi_{\text{i.c.}}(r,t) = \int_{x_1=0}^{L_1}\int_{x_2=0}^{L_2}\int_{x_3=0}^{L_3} \times G_\psi(r,t;r',0)E^{-1}(r',0)F(r')\,\mathrm{d}x_1\,\mathrm{d}_2\,\mathrm{d}x_3 \tag{23}$$

The Green's function $G_\psi(r,t;r',\tau) = G_\psi(x_1,x_2,x_3,t;x_1',x_2',x_3',\tau)$ is discussed further below. The volumetric energy generation term is

$$\psi_g(r,t) = \frac{1}{C}\int_{\tau=0}^{t}\int_{x_1=0}^{L_1}\int_{x_2=0}^{L_2}\int_{x_3=0}^{L_3} G_\psi(r,t;r',\tau) \times E^{-1}(r',\tau) g(r',\tau)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}\tau \quad (24)$$

The boundary condition term is

$$\psi_{\mathrm{b.c.}}(r,t) = \frac{1}{C}\int_{\tau=0}^{t}\sum_{\substack{\text{second,third}\\ \text{kinds b.c.}}}\int_{s_j} G_\psi(r,t;r'_j,\tau) \times E^{-1}(r'_j,\tau) f_j(r'_j,\tau)\,\mathrm{d}s_j\,\mathrm{d}\tau + \int_{\tau=0}^{t}\sum_{\substack{\text{first kind}\\ \text{b.c.}}}\int_{s_j}\frac{\kappa_i}{C}\left(-\frac{\partial G_\psi}{\partial n'_j}(r,t;r'_j,\tau)\right) \times E^{-1}(r'_j,\tau) f_j(r'_j,\tau)\,\mathrm{d}s_j\,\mathrm{d}\tau \quad (25)$$

where the summations on j go from 1 to 6 for the surfaces of a parallelepiped but the first summation includes only the boundary conditions of the second and third kinds and second summation includes just the first kind boundary conditions.

The Green's functions for ψ are those for Eq. (21a) with the homogeneous boundary conditions given by those selected from Eqs. (13)–(15).

4. Product relation for the Green's functions in a homogeneous parallelepiped

The Green's functions for ψ in an isotropic parallelepiped can be readily found from the 1D GFs using the product relation of [5, pp. 98–103]

$$G_{\psi:XIJYKLZMN}(x,y,z,t;x',y',z',\tau) = G_{\psi:XIJ}(x,t;x',\tau)G_{\psi:YKL}(y,t;y',\tau)G_{\psi:ZMN}(z,t;z',\tau) \quad (26)$$

For clarity we have used the notation of XIJ instead of X_1IJ and so on. The notation in Eq. (26) denotes the boundary conditions on the surfaces of the parallelepiped. For example, XIJ denotes the x-direction boundary conditions of the Ith kind at $x = 0$ and the Jth kind at $x = L_1$; I and J can go from 1 for prescribed temperature (boundary condition of the first kind), 2 for prescribed heat flux (second kind) and 3 for prescribed ambient temperature (third kind). Similarly YKL is for the y-direction with boundary conditions of Kth kind at $y = 0$ and Lth kind at $y = L_2$. Also ZMN is similar.

The boundary conditions of the first kind on opposite boundaries are the simplest because the transformation Eq. (3a) does not change the boundary conditions. As mentioned in Section 1, two expressions are available for the Green's functions [5, p. 481, 482]. For the $X11$ case (and similarly for the $Y11$ and $Z11$ cases) the "short" and "long" time expressions are

$$G^{\mathrm{S}}_{\psi:X11}(x_1,t;x'_1,\tau) = \sum_{n=-\infty}^{n=\infty}\left(K(2nL_1 + x_1 - x'_1, t-\tau) - K(2nL_1 + x_1 + x'_1, t-\tau)\right) \quad (27a)$$

$$G^{\mathrm{L}}_{\psi:X11}(x_1,t;x'_1,\tau) = \frac{2}{L_1}\sum_{m=1}^{\infty} \mathrm{e}^{-(m\pi)^2\frac{2\alpha(t-\tau)}{L_1^2}} \times \sin\left(m\pi\frac{x_1}{L_1}\right)\sin\left(m\pi\frac{x'_1}{L_1}\right) \quad (27b)$$

where

$$K(w,\theta) \equiv \frac{1}{\sqrt{4\pi\alpha\theta}}\mathrm{e}^{-\frac{w^2}{4\alpha\theta}} \quad (27c)$$

Eq. (27a) is for the short time form and has an "S" superscript on the G; this form comes from the Laplace transform; Eq. (27b) is the long time form with a "L" superscript and comes from separation of variables solution method. The words "short time" and "long time" are not particularly apt because actually what we are describing is really short and long values of

$$u \equiv t - \tau \quad (27d)$$

This quantity could be called a "convolution time" but for brevity, we coin the word "cotime". To be more precise, the new word "cotime" is used below to describe u.

For the same values of x_1, x'_1, t and τ, Eqs. (27a) and (27b) give the same numerical answer. However, the required number of terms to get desired accuracy can be quite different in the two expressions. For small values of dimensionless cotime $\alpha u/L_1^2$ such as less than 0.05 only a few terms are needed in Eq. (27a) and only a few are needed in Eq. (27b) for $\alpha u/L_1^2$ greater than 0.05. (What is meant by "few" is made clear below.) Conversely, for large values of $\alpha u/L_1^2$ such as 1, many terms are needed using Eq. (27a) and the same is true for Eq. (27b) for that dimensionless group going to zero. In the time-partitioning method both of these forms are used. Each is used in its most efficient region, resulting in only a few terms being needed in each summation. This is true for 1D, 2D and 3D cases, with and without solid body flow.

5. Product variations in initial conditions, volumetric energy generation and boundary conditions

Since the 3D Green's function for the present problem can be formed by a product of the 1D GFs, the Green's function solution equation can be written in a relatively simple manner for product forms of initial conditions, volumetric energy generation and boundary conditions. Due to space limitations this topic is explored briefly. Let the initial condition be given by a product of a function of x_1 by function of x_2 by a function of x_3

$$F(x_1,x_2,x_3)=F_0F_1(x_1)F_2(x_2)F_3(x_3) \tag{28a}$$

where F_0 is a scaling constant. Using Eq. (23) the result for the $\psi_{\text{i.c.}}$ function is

$$\psi_{\text{i.c.}}(x_1,x_2,x_3,t)=F_0\prod_{i=1}^{3}\int_{x_i'=0}^{L_i}E_i^{-1}(x_i',0)\times G_{\psi:(XIJ)_i}(x_i,t;x_i',0)F_i(x_i')\,\mathrm{d}x_i' \tag{28b}$$

For the volumetric generation term, the result is similar with now a product on space and time; the integration is over 3D space and time as well. For a boundary condition at $x_1=0$ of the second kind, that is, a q of the form

$$f_1(x_2,x_3,t)=f_{10}f_{12}(x_2)f_{13}(x_3)f_i(t) \tag{29a}$$

and the temperature equal to zero at $x_1=L_1$, the $X21$ Green's function is needed. For the $x_1=0$ surface being the only nonhomogeneous one, Eq. (25) yields

$$\psi_{\text{b.c.}}(x_1x_2,x_3,t)=\frac{f_{10}}{C}\int_{u=0}^{t}E_1^{-1}(0,t-u)G_{\psi:X21}(x,0,u)\times\prod_{i=2}^{3}\mathrm{IG}_{\psi:(XIJ)_i}(x_i,u)\,\mathrm{d}u \tag{29b}$$

$$\mathrm{IG}_{\psi:(XIJ)_i}(x_i,u)\equiv\int_{x_i'=0}^{L_i}E_i^{-1}(x_i',t-u)G_{\psi:(XIJ)_i}(x_i,x_i',u)f_{1i}(x_i')\,\mathrm{d}x_i' \tag{29c}$$

5.1. Green's function for ψ for the X21 and X31 cases

The ψ Green's functions for boundary conditions are changed for solid body flow. We noted above that the boundary condition of the second kind on T goes to the third kind on ψ. The resulting ψ short cotime GF (restricted to $\alpha u/L_1^2\leqslant 0.05$) is [6]

$$\begin{aligned}G^{\mathrm{S}}_{\psi:X31}(x,x',u)&\approx K(x-x',u)+K(x+x',u)-K(2L_1-x-x',u)\\&\quad-K(2L_1-x+x',u)-\frac{B_{1,\text{eff}}}{L_1}[H_0(x+x',u)\\&\quad-H_0(2L_1-x+x',u)]\end{aligned} \tag{30a}$$

where the function $H_0(z,u)$ is defined as

$$H_0(z,u)\equiv \mathrm{e}^{B_{1,\text{eff}}\frac{z}{L_1}+B_{1,\text{eff}}^2\frac{\alpha u}{L_1^2}}\mathrm{erfc}\left(\frac{z}{2L_i}\left[\frac{\alpha u}{L_1^2}\right]^{-1/2}+B_{1,\text{eff}}\left[\frac{\alpha u}{L_1^2}\right]^{1/2}\right) \tag{30b}$$

The GF for the $X21$ case for ψ is the $X31$ Green's function with the heat transfer coefficient, h, replaced by the effective one given by Eq. (19a) with $h_j=0$; this then gives for the $\psi:X21$ case with $B_{1,\text{eff}}=-P_1/2$.

The long cotime form of GF for the $X31$ case (and also the $X21$ case) is

$$\begin{aligned}G_{\psi:X31}(x,x',u)&=\mathrm{e}^{\beta_0^2\frac{\alpha u}{L_1^2}}\frac{X_0(x)X_0(x')}{N_0}+\sum_{m=1}^{\infty}\mathrm{e}^{-\beta_m^2\frac{\alpha u}{L_1^2}}\frac{X_m(x)X_m(x')}{N_m}\\X_0(x)&=\sinh\left(\beta_0\frac{L_1-x}{L_1}\right);\\X_m(x)&=\sin\left(\beta_m\frac{L_1-x}{L_1}\right),\quad m=1,2,\ldots\end{aligned} \tag{31a}$$

$$\begin{aligned}N_0&=\frac{L_1}{2}\frac{-B_{1,\text{eff}}-R_0^2}{R_0^2},\quad N_m=\frac{L_1}{2}\frac{B_{1,\text{eff}}+R_m^2}{R_m^2}\\R_0^2&=B_{1,\text{eff}}^2-\beta_0^2,\quad R_m^2=B_{1,\text{eff}}^2+\beta_m^2\end{aligned} \tag{31b}$$

The summation part of Eq. (31a) is obtained directly from [5, p. 502] or from [6]. The eigencondition for the β_m eigenvalues are obtained from

$$\tan(\beta_m)=-\beta_m/B_{1,\text{eff}} \tag{31c}$$

The first part of Eq. (31a), which we call the zeroth term, was difficult to find. A method of determining this term for the $\psi:X21$ case is now given.

The one-dimensional equation for finding the Green's function G_ψ is

$$\frac{\partial^2G_\psi}{\partial x^2}=\frac{1}{\alpha}\frac{\partial G_\psi}{\partial t},\quad 0<x<L_1,\ \ t>0 \tag{32a}$$

For the $X21$ case the homogeneous boundary conditions are (see Eqs. (13) and (14))

$$-k\frac{\partial G_\psi}{\partial x}(0,t)=\frac{UC}{2}G_\psi(0,t),\quad G_\psi(L_1,t)=0 \tag{32b}$$

The usual procedure is to let

$$G_\psi(x,t)=X(x)\Theta(t) \tag{32c}$$

Introducing Eq. (32c) into Eq. (32a) gives

$$\frac{X''(x)}{X(x)}=\frac{1}{\alpha}\frac{\Theta'(t)}{\Theta(t)}=\left(\frac{\beta}{L_1}\right)^2 \tag{33}$$

Unlike the usual procedure, a positive constant is chosen in Eq. (33). That is a key for finding the zeroth eigenvalue. The solution for $X(x)$ is

$$X(x)=C_1\sinh\left(\beta\frac{L_1-x}{L_1}\right)+C_2\cosh\left(\beta\frac{L_1-x}{L_1}\right) \tag{34}$$

The boundary conditions on $X(x)$ are obtained using Eq. (32b) to get

$$-\frac{\partial X}{\partial x}(0)=\frac{U_1}{2\alpha}X(0),\quad X(L_1)=0 \tag{35}$$

Using the condition at L_1 in Eq. (34) gives $C_2=0$. Using the first condition of Eq. (35) yields an eigencondition containing a hyperbolic tangent (a surprising result),

$$\tanh(\beta_0)=\frac{2}{Pe_1}\beta_0,\quad P_1=\frac{U_1L_1}{\alpha} \tag{36a, b}$$

which is similar to Eq. (31c). Notice that the Peclet number can be positive and negative. (For the $\psi : X31$ case, $P_1/2$ is replaced by the negative of $B_{1,\text{eff}}$.) (A reviewer pointed out that Eq. (36) can be found from Eq. (31c) using an imaginary β_0 in Eq. (31c).)

Solving the $\Theta(t)$ equation in Eq. (33) and combining with (32b) and (34) gives

$$G_{\psi 0}(x,t) = A_0 e^{\beta_0^2 \frac{\alpha t}{L_1^2}} \sinh\left(\beta_0 \frac{L_1 - x}{L_1}\right) = A_0 e^{\beta_0^2 \frac{\alpha t}{L_1^2}} X_0(x) \tag{37}$$

It can be shown that the eigenfunction in this equation and the ones in the second part of Eq. (31a) (namely, $\sin[\beta_m(L_1 - x)/L_1]$) are orthogonal over the region $0 < x < L_1$. The Green's function component corresponding to Eq. (37) can be found in several ways. One is to consider the initial temperature distribution of $\delta(x')$ with homogeneous boundary conditions and then to solve for A_0. After multiplying Eq. (37) by $\sinh[\beta_0(L_1 - x)/L_1]$ and integrating over x gives

$$\int_{x=0}^{L_1} \sinh\left(\beta_0 \frac{L_1 - x}{L_1}\right) \delta(x')\,dx = \sinh\left(\beta_0 \frac{L_1 - x'}{L_1}\right) = A_0 N_0 \tag{38a}$$

$$N_0 = \int_{x=0}^{L_1} \sinh\left(\beta_0 \frac{L_1 - x}{L_1}\right) \sinh\left(\beta_0 \frac{L_1 - x}{L_1}\right) dx = \frac{L_1}{2}\left[-1 + \frac{\sinh(2\beta_0)}{2\beta_0}\right] \tag{38b}$$

This equation then gives

$$A_0 = \frac{1}{N_0} \sinh\left(\beta_0 \frac{L_1 - x'}{L}\right) = \frac{X_0(x')}{N_0} \tag{39a}$$

Using this expression in Eq. (37) then gives the zeroth term in the Green's function in Eq. (31b) and completes this derivation. Using Eq. (36) and a trigonometric identity it can be shown for the $\psi : X21$ case that

$$N_{0,\psi:X21} = \frac{L}{2} \frac{P_1 - R_0^2}{R_0^2} \tag{39b}$$

which is used only for P_1 greater than two. See the below discussion.

6. Eigenvalues for $\psi : X21$ and $\psi : X31$ cases

Special attention is needed for the determination of the eigenvalues for the $\psi : X21$ and $\psi : X31$ cases. Eigenconditions for the $\psi : X21$ case are given by Eqs. (36a) and (31c), where the B_1 given by Eq. (19b) reduces to $-P_1/2$. See Fig. 1 for a plot of the left sides of Eq. (36a) and Eq. (31c), $\tanh(\beta)$ and $\tan(\beta)$, respectively. The right sides of Eqs. (36a) and (31c) form straight lines from the origin with a slope proportional to $2/P$. The tangent function keeps repeating while the hyperbolic tangent starts at zero and increases to the value of unity. In the common use of Eq. (31c) to find the eigenvalues, the Biot number is positive and the right side of this equation is negative. See the $P = -4$ curve in Fig. 1 which corresponds to $B = 2$ for $U = 0$. The first eigenvalue is the intersection of the first $\tan(\beta)$ curve and the $P = -4$ line, giving a known value of about 2.29 [4, p. 655]; see the "X" mark in Fig. 1. However, for $P = 4$ (a positive velocity in the x-direction) the intersection of the $-\beta/B_{1,\text{eff}} = \beta/2$ curve and the $\tan(\beta)$ curve is where $\tan(\beta)$ is positive; see the value of $\beta_1 \approx 4.3$ indicated by a circle in Fig. 1. For P less than 2 in value, the straight lines shown in Fig. 1 only intersect the hyperbolic tangent curve at $\beta_0 = 0$. Hence the zeroth order term is *not* present for $-\infty \leqslant P \leqslant 2$. The $P = 4$ line in Fig. 1 intersects the hyperbolic tangent about 1.9; see β_0 indicated by a circle. For P greater than 2 the zeroth term in Eq. (31a) makes a contribution, possibly the dominant one.

Plots of some eigenvalues for the $\psi : X21$ case are given in Fig. 2 as a function of the Peclet number. (The abscissa could also be $-2B + P$ for the $\psi : X31$ case.) The eigenvalues for $-\infty \leqslant P \leqslant 2$ are commonly tabulated [4,5,14] but those above $P = 2$ are not. The eigenvalues for $P > 2$ are surprising. The eigenvalue labeled β_1 (which comes from the tangent eigencondition) does not exist above $P = 2$. Instead β_0, coming from the hyperbolic condition, appears. Then the β_{i+1} curves on the left go to the β_i curves on the right for $i = 1, 2, \ldots$ The curves could be labeled such that the labeling does not change at $P = 2$ but that poses a problem because the β_0 numerical values can exceed β_1 if P is sufficiently large.

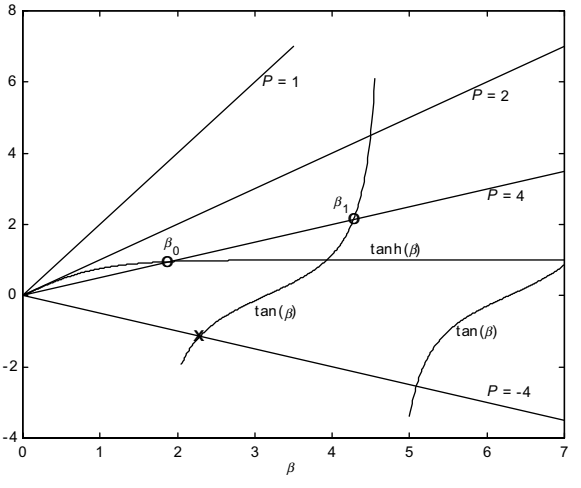

Fig. 1. Finding eigenvalues for the $\tan(\beta_m) = -\beta_m/B_{1,\text{eff}}$ and $\tanh(\beta_0) = 2\beta_0/P$ eigenconditions.

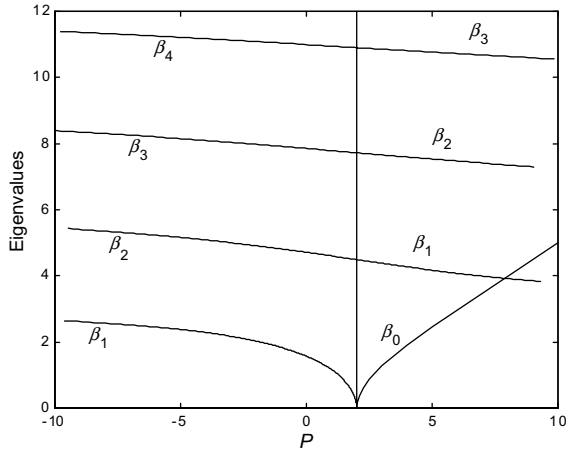

Fig. 2. First four eigenvalues ($\beta_1 - \beta_4$) as well as the zeroth eigenvalue (β_0) as functions of P. The zeroth eigenvalue only exists for $P > 2$.

7. Evaluation of Green's functions near the partition cotime

One way to validate the Green's functions expressions for the $\psi : X21$ case is to evaluate both of them at or below the partition cotime of $Fo = \alpha u/L_1^2 = 0.05$ at a given point. To get the long cotime values requires eigenvalues be calculated. For $P = 4$, the values are $\beta_0 \approx 1.915$, $\beta_1 \approx 4.275$, $\beta_2 \approx 7.597$ and then roughly $(2m-1)\pi/2$. For simplicity assume that the eigenvalues are temporarily approximated by $m\pi$; then to make last exponential term in the infinite series in Eq. (31a) less than 1.0E−10, let

$$e^{-m_{\max}^2 \pi^2 Fo} = 1.0E-10, \quad m_{\max} \approx \sqrt{23/(\pi^2 Fo)} \approx 1.5/\sqrt{Fo} \tag{40}$$

since $\exp(-23.025) = 1.001E-10$. Using the values of $Fo = 0.04$, 0.05 and 0.06 gives the maximum required number of terms of 7.5, 6.7 and 6.12, which round to 7, 6 and 6, respectively. The number of terms about six is very acceptable. In contrast, if the cotime Fo value is allowed to approach zero, Eq. (40) indicates that the number of terms increases without limit. Fortunately using the time partition method, the short cotime GF equation, Eq. (30a,b) is efficiently used for the smaller values of Fo.

8. Example

An example for the temperature distribution in a parallelepiped is given for flow only in the x-direction and the only nonhomogeneous boundary condition is a constant heat flux at $x = 0$. The initial temperature is zero. The temperature at $x_1 = L_1$ is zero. At this point, the boundary conditions at the y- and z-surfaces are homogenous and can be of the first, second or third kinds. The heat flux given by Eq. (29a) is a constant so that each function is equal to unity and $f_{10} = q_0$. Using the notation of [5], this 3D problem can be described by $XU21B10$ $YMNB00$ $ZKLB00T0$, where M, N, K and L can each be 1, 2 or 3. Using Eqs. (2), (3) and (29b) the temperature is given by

$$T(x,y,z,t) = e^{\frac{Ux}{2\alpha} - \frac{U^2 t}{4\alpha}} \psi_{\text{b.c.}}(x,y,z,t)$$

$$\psi_{\text{b.c.}}(x,y,z,t) = \frac{q_0}{C} \int_{\tau=0}^{t} G_{\psi:X21}(x,0,u) e^{\frac{U^2(t-u)}{4\alpha}} \text{IG}_{YMN}(y,u) \text{IG}_{ZKL}(z,u)\, du \tag{41}$$

where IG denotes a spatial integral of a Green's function such as

$$\text{IG}_{YMN}(y,u) = \int_{y'=0}^{L_2} G_{YMN}(y,y',u)\, dy' \tag{42}$$

For multi-dimensional analytical heat conduction problems we have found that the time partitioning method of solution is very powerful for nonhomogeneous boundary conditions. See [5–7]. In this method, the integrals such as in Eq. (41) are broken into two parts so that the short and long cotime type Green's functions can be used where each is the most efficient. Then Eq. (41) is written as

$$\begin{aligned}\psi_{\text{b.c.}}(x,y,z,t) &= \frac{q_0}{C} e^{\frac{U^2 t}{4\alpha}} \int_{u=0}^{t_p} G^{\text{S}}_{\psi:X21}(x,0,u) e^{-\frac{U^2 u}{4\alpha}} \text{IG}^{\text{S}}_{YMN}(y,u) \\ &\quad \times \text{IG}^{\text{S}}_{ZKL}(z,u)\, du + \frac{q_0}{C} e^{\frac{U^2 t}{4\alpha}} \int_{u=t_p}^{t} G^{\text{L}}_{\psi:X21}(x,0,u) e^{-\frac{U^2 u}{4\alpha}} \\ &\quad \times \text{IG}^{\text{L}}_{YMN}(y,u) \text{IG}^{\text{L}}_{ZKL}(z,u)\, du \end{aligned} \tag{43}$$

For simplicity at this point in this example, consider the problem where the boundary conditions of the second kind are at $y = 0$ and L_2 and $z = 0$ and L_3; in this case the integrals over the surface like Eq. (42) are unity and the problem becomes one-dimensional and Eq. (43) becomes

$$\begin{aligned}\psi_{\text{b.c.}}(x,t) &= \frac{q_0}{C} e^{\frac{U^2 t}{4\alpha}} \left[\int_{u=0}^{t_p} G^{\text{S}}_{\psi:X21}(x,0,u) e^{-\frac{U^2 u}{4\alpha}} du \right. \\ &\quad \left. + \int_{u=t_p}^{t} G^{\text{L}}_{\psi:X21}(x,0,u) e^{-\frac{U^2 u}{4\alpha}} du \right] \\ &= \psi^{\text{S}}_{\text{b.c.}}(x,t_p) + \psi^{\text{L}}_{\text{b.c.}}(x,t_p,t) \end{aligned} \tag{44}$$

Both the long and short cotime solutions are used in Eq. (44).

The first integral in Eq. (44) is evaluated using Eq. (30) at $x' = 0$ with $B_{\text{eff}} = -UL_1/(2\alpha) = -P/2$. This gives the short cotime GF,

$$G^{\text{S}}_{\psi:X21}(x,0',u) \approx 2K(x,u) - 2K(2L_1 - x,u) + \frac{P}{2L_1}[H_0(x,u) - H_0(2L_1 - x,u)] \qquad (45)$$

This expression can be written as two parts, one for x and the other for $2L_1 - x$; let z be either of these values. A typical integral for a component of the short cotime GF is then

$$\int_{u=0}^{t_{\text{p}}} \left\{ \frac{1}{\sqrt{\pi\alpha u}} \text{e}^{-\frac{z^2}{4\alpha u} - \frac{U^2 u}{4\alpha}} - \frac{(-P/2)}{L_1} \text{e}^{-\frac{P}{2L_1}z + \left(\frac{P}{2}\right)^2 \frac{\alpha u}{L_1^2} - \frac{U^2}{4\alpha}u} \text{erfc}\left(\frac{z}{2L_1}\left[\frac{\alpha u}{L_1^2}\right]^{-1/2} + \left(-\frac{P}{2}\right)\left[\frac{\alpha u}{L_1^2}\right]^{1/2} \right) \right\} \text{d}u$$
$$= \frac{L_1}{\alpha} \int_{w=0}^{Fo_{\text{p}}} \left\{ \frac{1}{\sqrt{\pi w}} \text{e}^{-a^2 w - \frac{b^2}{w}} - a\text{e}^{2ab} \text{erfc}\left[a\sqrt{w} + \frac{b}{\sqrt{w}} \right] \right\} \text{d}w$$
$$= \frac{L_1}{\alpha} I_{\psi 21}(a,b,Fo_{\text{p}}) \qquad (46a)$$

where Appendix A is used and

$$a = -\frac{P}{2}, \quad b = \frac{z}{2L_1}, \quad w = \frac{\alpha u}{L_1^2}, \quad Fo = \frac{\alpha t}{L_1^2} \qquad (46b)$$

Then the short cotime component in Eq. (44) is used with Eqs. (3) and (46a) to get the short cotime temperature component for the $XU21B10T0$ problem,

$$T^{\text{S}}(x,t_{\text{p}}) = \frac{q_0 L_1}{k} \text{e}^{\frac{P}{2L_1}x} \left[I_{\psi 21}\left(-\frac{P}{2}, \frac{x}{L_1}, Fo_{\text{p}} \right) - I_{\psi 21}\left(-\frac{P}{2}, \frac{2L_1 - x}{L_1}, Fo_{\text{p}} \right) \right] \qquad (47)$$

Next consider the long cotime component of the temperature. Using Eq. (31a) and the long cotime component of Eq. (44) gives

$$T^{\text{L}}(x,t_{\text{p}},t) = \text{e}^{\frac{Ux}{2\alpha} - \frac{U^2 t}{4\alpha}} \psi^{\text{L}}_{\text{b.c.}}(x,t_{\text{p}},t) = T^{\text{L}}_{c.t.}(x,t) - T^{\text{L}}_{c.t.}(x,t_{\text{p}}) \qquad (48a)$$

where

$$T^{\text{L}}_{c.t.}(x,u) = -\frac{q_0 L_1}{k} \text{e}^{\frac{P}{2L_1}x} \left[\text{e}^{-R_0^2 \frac{\alpha u}{L_1^2}} \frac{X_0(x)X_0(0)}{L_1 N_0 R_0^2} + \sum_{m=1}^{\infty} \text{e}^{-R_m^2 \frac{\alpha u}{L_1^2}} \frac{X_m(x)X_m(0)}{L_1 N_m R_m^2} \right] \qquad (48b)$$

and the eigenfunctions X_0 and X_m are given in Eq. (31a). Because expressions similar to the right side of Eq. (48b) occur frequently in separation of variable GF solutions, it is proposed that the right side of Eq. (48b) be termed the "*c*omplementary *t*ransient" and denoted by the subscripts of "*c.t.*" Again note that the zeroth term is present only for $P > 2$ for the $\psi : X21$ problem.

The temperature using the notation in Eqs. (47) and (48a) gives the temperature at time t as

$$T(x,t) = T^{\text{S}}(x,t_{\text{p}}) + T^{\text{L}}(x,t_{\text{p}},t) = T^{\text{S}}(x,t_{\text{p}}) + T^{\text{L}}_{c.t.}(x,t) - T^{\text{L}}_{c.t.}(x,t_{\text{p}}) \qquad (49)$$

Notice that the steady state solution is given simply by

$$T(x) = T(x,\infty) = T^{\text{S}}(x,t_{\text{p}}) - T^{\text{L}}_{c.t.}(x,t_{\text{p}}) \qquad (50)$$

In words this equation states that the steady state temperature is the short cotime solution minus the complementary transient solution, both evaluated at the same partition cotime. This result applies for two- and three-dimensional problems as well as for 1D. Eqs. (49) and (50) contain implicitly the possibility of internal verification of the solution, which is briefly discussed next.

9. Tables and numerical values

Table 1 shows results for $P = 4$ using several different dimensional partition times for the above steady state part of the above problem, which is denoted $XU21B10$. The partition time method is not needed for this case since the steady state is simply [8]

$$T(x)/(q_0 L/k) = \frac{1}{P}\left(\text{e}^{P} - \text{e}^{P\frac{x}{L}}\right) \qquad (51)$$

However, this example is convenient to demonstrate the power of the time partitioning procedure. Some multidimensional examples for non-flow problems are given elsewhere [5–7]. Dimensionless partition cotimes used in Table 1 vary from 0.04 to 0.25 and the absolute errors vary in magnitude from 4E−15 to 1E−3, which vary from 3E−14 to 0.01%. It is important to note that the accuracy is to about 1 part in 10^{10} for any dimensionless partition cotime equal to or less than 0.05 in this example; we frequently use this value for the dimensionless partition cotime. As the cotime becomes smaller, the number of terms in the series increases. Note that increasing the number of terms from 3 to 7 at $x = 0$ decreases the errors from about 1.4E−3 to 6.6E−14, which is about ten orders of magnitude improvement with a just a doubling in the computations! This extreme accuracy with only a moderate computational load is a very attractive feature of these exact solutions with time partitioning.

The small numbers of terms mentioned above also exists for each of the double summations for 2D cases or triple summations for 3D cases.

The one-dimensional transient solution where the left boundary condition is of a prescribed heat flux and the

Table 1
Results for dimensionless temperature for $P = 4$ using several dimensional partition cotimes for the steady state case denoted $XU21B10$

Fo_p	x/L	Number of terms	$T(x/L)/q_0L/k$	Error
0.04	0.0	7	13.3995375083	6.57E−14
0.05	0.0	6	13.3995375083	1.33E−11
0.06	0.0	6	13.3995375088	4.93E−10
0.07	0.0	5	13.3995375150	6.75E−09
0.08	0.0	5	13.3995375577	4.94E−08
0.25	0.0	3	13.4009780223	1.44E−03
0.05	0.5	6	11.8022734836	−3.55E−15
0.08	0.5	5	11.8022734836	7.24E−11
0.1	0.5	4	11.8022734886	5.02E−09
0.12	0.5	4	11.8022735725	8.90E−08
0.25	0.5	3	11.8025046823	2.31E−04

This table shows the error associated with the selection of various cotimes.

right side is of a prescribed temperature condition, denoted $XU21B10T0$, is given by Eq. (49) with components from Eqs. (47) and (48). Some numerical values for the temperature and heat flux are given in Table 2 for $P = 1$ (no zeroth term present) and in Table 3 for $P = 10$ (which has a zeroth term, as indicated by the eigenvalues given in the titles). Notice that the Table 2 cases come to a steady state condition within a dimensionless time of about 4 with a value of 0.5 for x/L. The steady state surface temperature is given as 1.7182818285. The results in Table 3, for $P = 10$, are given for a large dimensionless time in order to reach a steady state in this case. For the heated surface, the temperature at dimensionless time 10,000 equals the steady state temperature to 12 digits. The duration to reach a steady state is extremely long in comparison to the solution for $P = 0$.

Many of the heat fluxes shown in Table 3 are extremely large for $x > 0$. At first the values seem physically unreasonable, going to over 13,000 times the surface heat flux. Two suggestions are made regarding this result. First, the solution can be used as a manufactured solution for verification purposes. Second, the solution might possibly lend some insight into the behavior of turbulent heat transfer as modeled by surface renewal theory [15].

Table 2
Dimensionless temperatures and heat fluxes using the short and long cotime expressions with $P = 1$

Fo	x^+	$\frac{T(x,t)}{q_0L/k}$	$\frac{q(x,t)}{q_0}$
0.001	0	0.0361854558	1.0000000000
0.01	0	0.1179319248	1.0000000000
0.1	0	0.4097894342	1.0000000000
0.2	0	0.6123851323	1.0000000000
0.4	0	0.9175253431	1.0000000000
1	0	1.4132503381	1.0000000000
2	0	1.6572202925	1.0000000000
4	0	1.7182818285	1.0000000000
0.001	0.5	0.0000000000	0.0000000000
0.01	0.5	0.0000187012	0.0005213862
0.1	0.5	0.0826505020	0.3348179150
0.2	0.5	0.2214341992	0.5657252009
0.4	0.5	0.4538600983	0.8677404431
1	0.5	0.8350106160	1.3512463891
2	0.5	1.0226080950	1.5891724249
4	0.5	1.0695605578	1.6487212707

The first three eigenvalues are: $\beta_1 = 1.1655611852$, $\beta_2 = 4.6042167772$ and $\beta_3 = 7.7898837511$.

Table 3
Dimensionless temperatures and heat fluxes for $P = 10$

Fo	x^+	$\frac{T(x,t)}{q_0L/k}$	$\frac{q(x,t)}{q_0}$
0.01	0	0.17201411	1.0000000000
0.1	0	1.09943616	1.0000000000
1	0	10.08393416	1.0000000000
10	0	97.93266295	1.0000000000
100	0	804.32890105	1.0000000000
1000	0	2179.12596424	1.0000000000
10000	0	2202.54657948	1.0000000000
0.1	0.5	0.61025967	0.9310800731
1	0.5	9.52822942	1.5819725276
10	0.5	96.80856180	7.4652805581
100	0.5	798.63429449	54.7732691585
1000	0.5	2164.53618375	146.8446595284
10000	0.5	2187.80526357	148.4131591026
0.1	0.95	0.13397607	2.1210447835
1	0.95	3.65583214	56.4192513748
10	0.95	38.23568692	589.4630694768
100	0.95	316.29411377	4875.6963823107
1000	0.95	857.45485049	13217.6164731432
10000	0.95	866.67389651	13359.7268296619

The first few eigenvalues are $\beta_0 = 4.9995456086$, $\beta_1 = 3.7902223783$, $\beta_2 = 7.2502483071$, $\beta_3 = 10.55311042990$.

10. Summary and Conclusions

This paper gives a derivation for exact solutions of the transient orthotropic heat conduction equation for solid body flow in a plate, rectangle or parallelepiped. An objective is to develop a general method for providing extremely accurate and efficient solutions for verification of large multi-dimensional computer codes using finite element, control volume and other related approximate methods. Two transformations are used to obtain the standard form of the transient heat conduction equation which does not contain the flow terms and is for isotropic bodies. Green's functions (GFs) are available for this standard equation.

After developing the transformed equation, boundary and initial conditions, two types of one-dimensional GFs are used, one comes from the Laplace transform and the other from separation of variables. The multi-dimensional GFs are simply formed as a product of the 1D GFs. Each component of the 3D Green's functions is a function of $u = t - \tau$, for which we propose the name of "cotime". The Laplace transform-base GFs are used for the short cotimes and separation of variables-type are used for the long cotimes. These two forms are of the GFs are complementary, providing a very efficient method of solution. Furthermore, use of these two forms contains an internal verification capability.

The solid body motion changes the eigenvalues for the long cotime GFs. Moreover, the heat flux boundary condition introduces a previously unknown zeroth eigenvalue and function. The paper ends with an example which contains this eigenvalue.

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Appendix A. Derivation of a short cotime integral for $\psi : X21$ case

Consider the integral

$$I_{\psi 21}(a,b,Fo) = \int_{w=0}^{Fo} \left\{ \frac{1}{\sqrt{\pi w}} e^{-a^2 w - \frac{b^2}{w}} - a e^{2ab} \operatorname{erfc}\left[a\sqrt{w} + \frac{b}{\sqrt{w}} \right] \right\} dw \tag{A.1}$$

Using Eq. (12), p. 428 of [5], the first part of Eq. (A.1) is equal to

$$\begin{aligned} I_1(a,b,Fo) &= \int_{w=0}^{Fo} \frac{1}{\sqrt{\pi w}} e^{-a^2 w - \frac{b^2}{w}} dw \\ &= -\frac{1}{2a} \left[e^{2ab} \operatorname{erfc}\left(a\sqrt{w} + \frac{b}{\sqrt{w}} \right) + e^{-2ab} \operatorname{erfc}\left(a\sqrt{w} - \frac{b}{\sqrt{w}} \right) \right]_0^{Fo} \end{aligned} \tag{A.2}$$

The second part is [5, Eq. (1), p. 426]

$$\begin{aligned} I_2(a,b,Fo) &= \int_{w=0}^{Fo} a e^{2ab} \operatorname{erfc}\left[a\sqrt{w} + \frac{b}{\sqrt{w}} \right] dw \\ &= \left[-\sqrt{\frac{w}{\pi}} e^{-a^2 w - \frac{b^2}{w}} - \frac{1}{4a} e^{-2ab} \times \operatorname{erfc}\left(a\sqrt{w} - \frac{b}{\sqrt{w}} \right) + \left(aw + b - \frac{1}{4a} \right) e^{2ab} \times \operatorname{erfc}\left(a\sqrt{w} + \frac{b}{\sqrt{w}} \right) \right]_0^{Fo} \end{aligned} \tag{A.3}$$

Using these last two equation in Eq. (A.1) and introducing the limits gives

$$\begin{aligned} I_{\psi 21}(a,b,Fo) &= \sqrt{\frac{Fo}{\pi}} e^{-a^2 Fo - \frac{b^2}{Fo}} - e^{2ab} \operatorname{erfc}\left[a\sqrt{Fo} + \frac{b}{\sqrt{w}} \right] \left(aFo + b + \frac{1}{4a} \right) \\ &\quad + \frac{1}{4a} e^{-2ab} \operatorname{erfc}\left[-a\sqrt{Fo} + \frac{b}{\sqrt{w}} \right] \end{aligned} \tag{A.4}$$

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